NASA-CR-200984

Visualization of 2-D and 3-D Tensor Fields

Principal Investigator: Lambertus Hesselink*
Stanford University, Stanford, California 94305-4035
Phone: (415)723-4850 Email: bert@kaos.stanford.edu

Introduction

In previous work we have developed a novel approach to visualizing second order symmetric 2-D tensor fields based on degenerate point analysis. At degenerate points the eigenvalues are either zero or equal to each other, and the hyperstreamlines about these points give rise to trisector or wedge points. These singularities and their connecting hyperstreamlines determine the topology of the tensor field. In this study we are developing new methods for analyzing and displaying 3-D tensor fields. This problem is considerably more difficult than the 2-D one, as the richness of the data set is much larger. Here we report on our progress and a novel method to find , analyze and display 3-D degenerate points. First we discuss the theory, then an application involving a 3-D tensor field, the Boussinesq problem with two forces.

Definitions

Second-Order Tensor field

Let $\mathcal{L}(X,Y)$ be the set of all the linear transformations of the vector space X into the vector space Y, and let E be an open subset of $\mathbf{R}^n$. A second-order tensor field $\mathbf{T}(\vec{x})$ defined across E is a mapping: $\mathbf{T} : E \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$ that associates to each vector $\vec{x}$ of E a linear transformation of $\mathbf{R}^m$ into itself. If $\mathbf{R}^m$ is referenced by a Cartesian coordinate system, $T(\vec{x})$ can be represented

*Professor, Electrical Engineering and Aeronautics and Astronautics Departments.

by m^2 Cartesian components $T_{ij}(\vec{x}), i, j = 1, ..., m$ that transform according to

$$T'_{ij} = \sum_{p,q=1}^{m} \beta_{ip}\beta_{jq}T_{pq} \tag{1}$$

under an orthonormal transformation $\beta = \{\beta_{ij}\}$ of the coordinate axes.[1]

2-D and 3-D tensor fields can be represented by 2×2 and 3×3 matrices respectively. In Cartesian coordinates the matrices representing symmetric tensor fields take the following forms:

- 2-D symmetric tensor fields

$$\mathbf{T}(\vec{x}) = \mathbf{T}(x, y) = \begin{pmatrix} T_{11}(x,y) & T_{12}(x,y) \\ T_{12}(x,y) & T_{22}(x,y) \end{pmatrix} \tag{2}$$

- 3-D symmetric tensor fields

$$\mathbf{T}(\vec{x}) = \mathbf{T}(x, y, z) = \begin{pmatrix} T_{11}(x,y,z) & T_{12}(x,y,z) & T_{13}(x,y,z) \\ T_{12}(x,y,z) & T_{22}(x,y,z) & T_{23}(x,y,z) \\ T_{13}(x,y,z) & T_{23}(x,y,z) & T_{33}(x,y,z) \end{pmatrix} \tag{3}$$

Degenerate Points

A degenerate point of a tensor field $\mathbf{T} : E \rightarrow \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$, where E is an open subset of $\mathbf{R}^m$, is a point $\vec{x}_0 \in E$ where at least two of the m eigenvalues of $\mathbf{T}$ are equal to each other.

In the case of 2-D tensor fields, $\vec{x}_0$ is a degenerate point iff $\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0)$. For 3-D tensor fields, various types of degenerate points exist, corresponding to the conditions $\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0)$, $\lambda_2(\vec{x}_0) = \lambda_3(\vec{x}_0)$, or $\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0) = \lambda_3(\vec{x}_0)$.

Locating Degenerate Points

- 3-D tensor

The first thing to note about a 3-D symmetric tensor field (Eq. (3)) is that it has 6 independent variables, three of which are on its diagonal. As a

result, various types of degenerate points may exist. These types correspond to the following conditions:

$$\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0) > \lambda_3(\vec{x}_0) \tag{4}$$

$$\lambda_1(\vec{x}_0) > \lambda_2(\vec{x}_0) = \lambda_3(\vec{x}_0) \tag{5}$$

$$\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0) = \lambda_3(\vec{x}_0) \tag{6}$$

A degenerate point that corresponds to the condition described by Eq. (6) implies that the tensor at that location always appears in its diagonalized form:

$$\mathbf{T}(\vec{x}_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{7}$$

The current method for locating this type of degenerate point involves solving 5 simultaneous equations.

$$\begin{cases} T_{11}(\vec{x}_0) - T_{22}(\vec{x}_0) = 0 \\ T_{22}(\vec{x}_0) - T_{33}(\vec{x}_0) = 0 \\ T_{12}(\vec{x}_0) = 0 \\ T_{13}(\vec{x}_0) = 0 \\ T_{23}(\vec{x}_0) = 0 \end{cases} \tag{8}$$

Since we are working in a 3-D space, grid cells are cubes (Fig. (1)). Let f, g, p, q and r be the values of $T_{11} - T_{22}$, $T_{22} - T_{33}$, T_{12}, T_{13} and T_{23} at the vertex (i, j, k) and f_{max}, f_{min}, g_{max}, g_{min}, p_{max}, p_{min}, q_{max}, q_{min}, r_{max}, r_{min} be the maximum and minimum values at these eight vertices respectively.

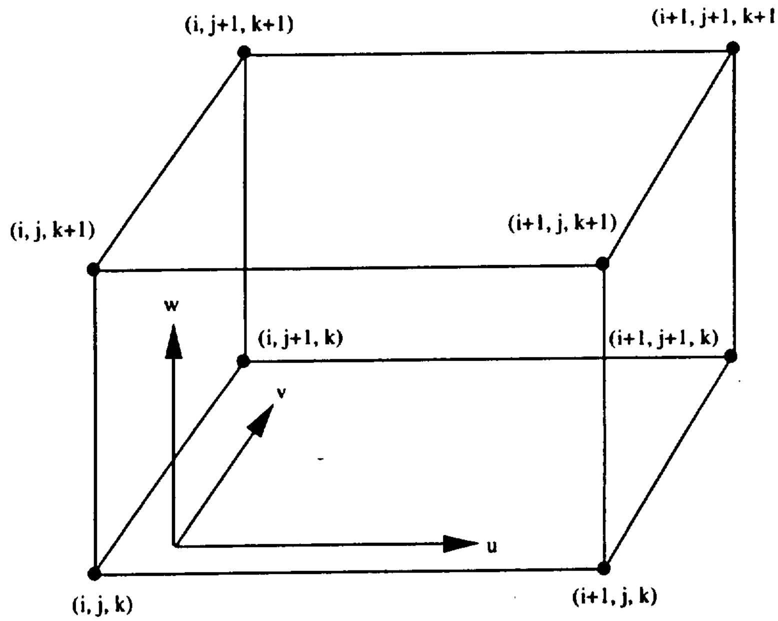

Figure 1: A 3-D grid cell.

Only when all of the following conditions:

$$\left\{\begin{array}{l} f_{min} \leq 0 \\ f_{max} \geq 0 \\ g_{min} \leq 0 \\ g_{max} \geq 0 \\ p_{min} \leq 0 \\ p_{max} \geq 0 \\ q_{min} \leq 0 \\ q_{max} \geq 0 \\ r_{min} \leq 0 \\ r_{max} \geq 0 \\ |f_{min}| + |f_{max}| > 0 \\ |g_{min}| + |g_{max}| > 0 \\ |p_{min}| + |p_{max}| > 0 \\ |q_{min}| + |q_{max}| > 0 \\ |r_{min}| + |r_{max}| > 0 \end{array}\right. \tag{9}$$

are met, will there possibly be a degenerate point inside the cell. Then the next step is to interpolate f, g, p, q, r to approximate the tensor within the cell. In order to do the linear interpolation , we need to do the first order approximation for all of the f, g, p, q, r which gives us 5 corresponding

variables s, t, u, v, w. At this moment the physical meaning associated with the two extra variables s and t is unclear, where as the other 3 parameters represent the local position variables u, v, w.

We can readily see that merely checking whether or not all 15 conditions are satisfied for each cube is a lot of work. Also, we always have to solve 5 nonlinear equations simultaneously. Furthermore, this method cannot be applied to degenerate points that correspond to the conditions described by Eqs. (4,5). In these cases the tensor may look like:

$$\mathbf{T}(\vec{x}_0) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \tag{10}$$

or

$$\mathbf{T}(\vec{x}_0) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \tag{11}$$

However, these tensors will be diagonal matrices only when they are in their eigenvector space. More generally, in any other spaces, they will take on the form:

$$\mathbf{T}(\vec{x}) = \begin{pmatrix} T_{11}(\vec{x}) & T_{12}(\vec{x}) & T_{13}(\vec{x}) \\ T_{12}(\vec{x}) & T_{22}(\vec{x}) & T_{23}(\vec{x}) \\ T_{13}(\vec{x}) & T_{23}(\vec{x}) & T_{33}(\vec{x}) \end{pmatrix} \tag{12}$$

A continuous tensor field varies at different locations, but we only have sampled data at grid points. Therefore we won't be able to know the eigenvector space of a field point off the grid, we usually choose a space by interpolating grid points in its neighborhood, but the tensor will appear just as a regular tensor. It is very difficult to find a simple rule to check its double degeneracy that will work in any random space.

Because of these difficulties, we went back to the original tensor and looked at it in a different way.

Since a second-order tensor representation is a matrix, we can get its eigenvalues by solving its characteristic equation.

- 3-D case

A 3-D symmetric tensor field can be expressed as a 3×3 matrix(Eq. (3)) and its characteristic equation is:

$$A(\lambda) = -\lambda^3 + a\lambda^2 + b\lambda + c \tag{13}$$

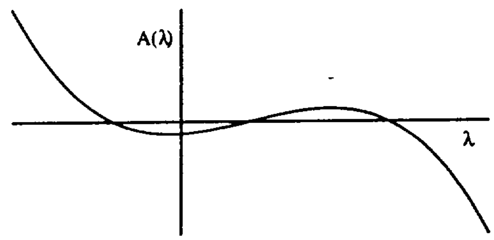

Figure 2: A typical three-root polynomial of order 3.

where

$$
\begin{array}{l}
a = T_{11} + T_{22} + T_{33} \\
b = T_{12}^2 + T_{13}^2 + T_{23}^2 - T_{11}T_{22} - T_{11}T_{33} - T_{22}T_{33} \\
c = T_{11}T_{22}T_{33} + 2T_{12}T_{13}T_{23} - T_{11}T_{23}^2 - T_{22}T_{13}^2 - T_{33}T_{12}^2
\end{array} \tag{14}
$$

and it's derivative is:

$$\frac{dA(\lambda)}{d\lambda} = -3\lambda^2 + 2a\lambda + b \tag{15}$$

The characteristic equation $A(\lambda)$ is a polynomial of order three and as such it can be generally plotted as in Fig. (2). The number of roots represents the number of different eigenvalues at each location: three-roots means non-degeneracy, two-roots means double degeneracy, one-root means triple degeneracy.

The various possible cases of degeneracy are determined by the solutions to Eq.(15).

$$\lambda_{i,ii} = \frac{a \pm \sqrt{d}}{3} \tag{16}$$

where

$$d = a^2 + 3b \tag{17}$$

The three possible degenerate cases are as follows:

1. $\lambda_i = \lambda_1 = \lambda_2 > \lambda_3$

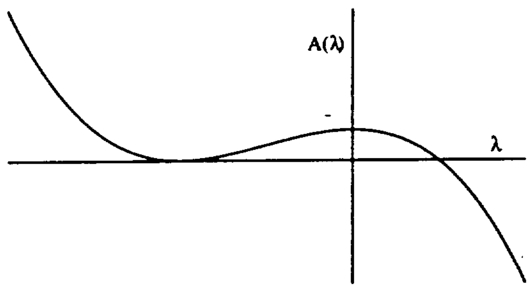

Figure 3: Two-root polynomial of order 3.

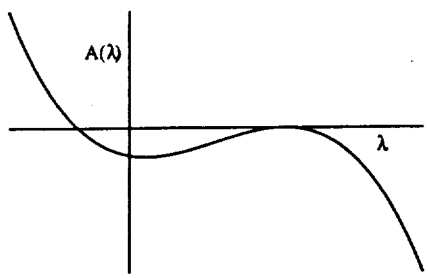

Figure 4: Two-root polynomial of order 3.

$$A(\lambda_i) = 0 \tag{18}$$

2. $\lambda_1 > \lambda_2 = \lambda_3 = \lambda_{ii}$

$$A(\lambda_{ii}) = 0 \tag{19}$$

3. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_i = \lambda_{ii}$

$$A(\lambda_i = \lambda_{ii}) = 0 \tag{20}$$

Also,

$$d(\lambda_i = \lambda_{ii}) = a^2 + 3b = 0 \tag{21}$$

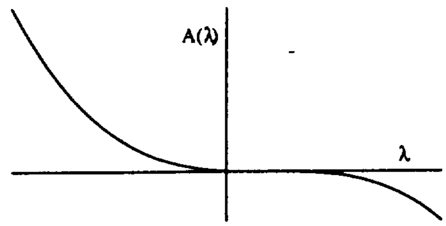

Figure 5: One-root polynomial of order 3.

After the solutions to the roots of the derivative equation being substituted into the characteristic equation, we get three spatial functions:

$$B_1(x,y,z) = A(\lambda_i) = \frac{2a^3 + 9ab + d^{3/2}}{27} + c \tag{22}$$

$$B_2(x,y,z) = A(\lambda_{ii}) = \frac{2a^3 + 9ab - d^{3/2}}{27} + c \tag{23}$$

$$B_3(x,y,z) = d(\lambda_i = \lambda_{ii}) = a^2 + 3b \tag{24}$$

These are the conditions for both double degeneracy and triple degeneracy. Our problem has been reduced to finding points where the functions $B_1(x,y,z) = 0$, $B_2(x,y,z) = 0$, or $B_3(x,y,z) = 0$.

Locating the Singular Points

In 3-D space, $B_1(x,y,z) = 0$ is a maximum for $B_1(x,y,z)$, $B_2(x,y,z) = 0$ is a minimum for $B_2(x,y,z)$ and $B_3(x,y,z) = 0$ is a maximum for $B_3(x,y,z)$.

Because the data around the extrema don't change sign, it is a challenge to find these points. Fortunately, the gradient components around extrema does change sign. Let $F(x,y,z)$ be a 3-D function. Let (x_0, y_0, z_0) be an

extremum of F then the following conditions exist:

$$\begin{array}{l} \frac{\partial F(x_0,y_0,z_0)}{\partial x} = 0 \\ \frac{\partial F(x_0,y_0,z_0)}{\partial y} = 0 \\ \frac{\partial F(x_0,y_0,z_0)}{\partial z} = 0 \end{array} \tag{25}$$

Our problem has been then reduced to finding the points that satisfy one of the following conditions:

$$\begin{array}{l} \vec{\nabla} B_1 = 0 \\ \vec{\nabla} B_2 = 0 \\ \vec{\nabla} B_3 = 0 \end{array} \tag{26}$$

Marching cube[2] is a high-resolution 3D surface construction algorithm. It is very helpful for us to locate zero gradient points. First, we generate isosurfaces of:

$$\begin{array}{l} S1 = \frac{\partial F(x,y,z)}{\partial x} = 0 \\ S2 = \frac{\partial F(x,y,z)}{\partial y} = 0 \\ S3 = \frac{\partial F(x,y,z)}{\partial z} = 0 \end{array} \tag{27}$$

then, we derive the intersections of these three surfaces.

Now we find the locations where the gradients of $B_1(x,y,z)$, $B_2(x,y,z)$ and $B_3(x,y,z)$ are zero, but it is still possible that these places are just a local maximum or minimum for B_1, B_2 and B_3. We check them by substituting these values back into B_1, B_2 and B_3. If these functions are actually zero, then they are the real points.

Application And Future Work

Application

As an application, we studied the Boussinesq problems with 2 forces. The results are quite encouraging; we found points of both double and triple degeneracy. All points of triple degeneracy are isolated from one another; however, the lines and surfaces formed from the points of double degeneracy extend to these isolated points of triple degeneracy. Although we studied only one case, our findings point to the possibility that points of double degeneracy in 3-D space can't be isolated. Since the points of triple degeneracy are what we are really interested in, we then analyze the topological structure at a small neighborhood surrounding a degenerate point where

$\lambda_1 = \lambda_2 = \lambda_3$. We use hyperstreamlines to represent our eigenvector fields. In the figures shown below, forces are acting along the x-coordinate direction. (See Plate 1.) The tensor field surrounding the degenerate point is represented by hyperstreamlines.[1] The hyperstreamlines are integrated along the major eigenvector where the cross section is comprised of the two transverse eigenvalues. A typical cross section is an ellipse with the major and minor axes scaled by the transverse eigenvalues. When the eigenvalues of the corresponding transverse eigenvectors are not zero, the cross section is an ellipse and thus the hyperstreamlines appear as a tube; when one of the eigenvalues is zero, the cross section reduces to a line and if it remains zero along the trajectory the hyperstreamlines will appear as a ribbon; when both of the eigenvalues are equal, the cross section becomes a circle (this is a particular double degeneracy case) and if they remain equal along the trajectory then the hyperstreamline follows a line of double degeneracy. The color of the hyperstreamlines encode the magnitude of the major eigenvalue. If we are very close to the degenerate point, the value changes dramatically during a very short distance, therefore we linearize the log of the eigenvalues in our color mapping (except for Plate 1, which is taken from previous work).

Plates 1 through 4 present the preliminary results from a Boussinesq problem with two forces.

Plate 1 is a global view of hyperstreamlines of the major eigenvector field. Frame box dimensions range from -2.0 to 2.0 in each direction. Forces (represented by two red arrows) are acting at $(-1.0, 0.5, 0.0)$ and $(-1.0, -0.5, 0.0)$ and are pointing toward $+x$ direction. The two points of triple degeneracy (represented by two red dots) are at $(1.05, 0.0, 0.5)$ and $(1.05, 0.0, -0.5)$. Color encodes the magnitude of the major eigenvalues.

Plates 2 through 4 are three different points of view of an enlarged area around the degenerate point $(1.05, 0.0, 0.5)$ in Plate 1, where x ranges from 1.042 to 1.062, y ranges from -0.008 to 0.012 and z ranges from 0.492 to 0.512.

From Plate 2, we can see that the cross section of a hyperstreamline along the x direction (the same direction as the force) is a circle which means the two transverse eigenvalues are equal.

From Plates 2 and 3, we can see that a hyperstreamline in the x direction propagates along the force first, then turns right before the triple degenerate point. The hyperstreamlines behind the degenerate point propagate in the planes perpendicular to the force where one of the other two eigenvalues is

zero.

The pattern of the hyperstreamlines shown above represents only one type of triple degeneracy. The major eigenvectors propagate along the forces until they are very close to the degenerate point, then they all stay in the planes perpendicular to the forces. The pattern of their propagation in these planes is very similar to hyperstreamlines in a 2-D tensor field with a trisector point. The separatrix with the same direction as two forces is also a line comprised of points of double degeneracy for the other two eigenvalues, therefore the cross section of the hyperstreamline along this separatrix is a circle. The hyperstreamlines that propagate in the planes perpendicular to the forces are more like ribbons because one of the medium or minor eigenvalues are zeros in those areas.

Future Work

The results presented in the previous section reveal only a small part of the complex nature of 3-D tensor fields. The research planned for the coming year, based on the preliminary findings and in accordance with the original proposal, will focus on:

1. Extension of topological rules from 2-D to 3-D tensor fields.
2. Identification and classification of various types of points of double and triple degeneracy.
3. Developing technique to locate continuous lines of double degeneracy.
4. Studying the various topological features, i.e. lines and surfaces of double degeneracy, points of triple degeneracy, and the relations between them.

References

[1] Delmarcelle, T., "The Visualization of Second-Order Tensor Fields," Ph.D. Thesis, Stanford University, 1994.

[2] Lorensen, W. and Cline, H. "Marching Cubes: A High Resolution 3D Surface Construction Algorithm."

Plate 1

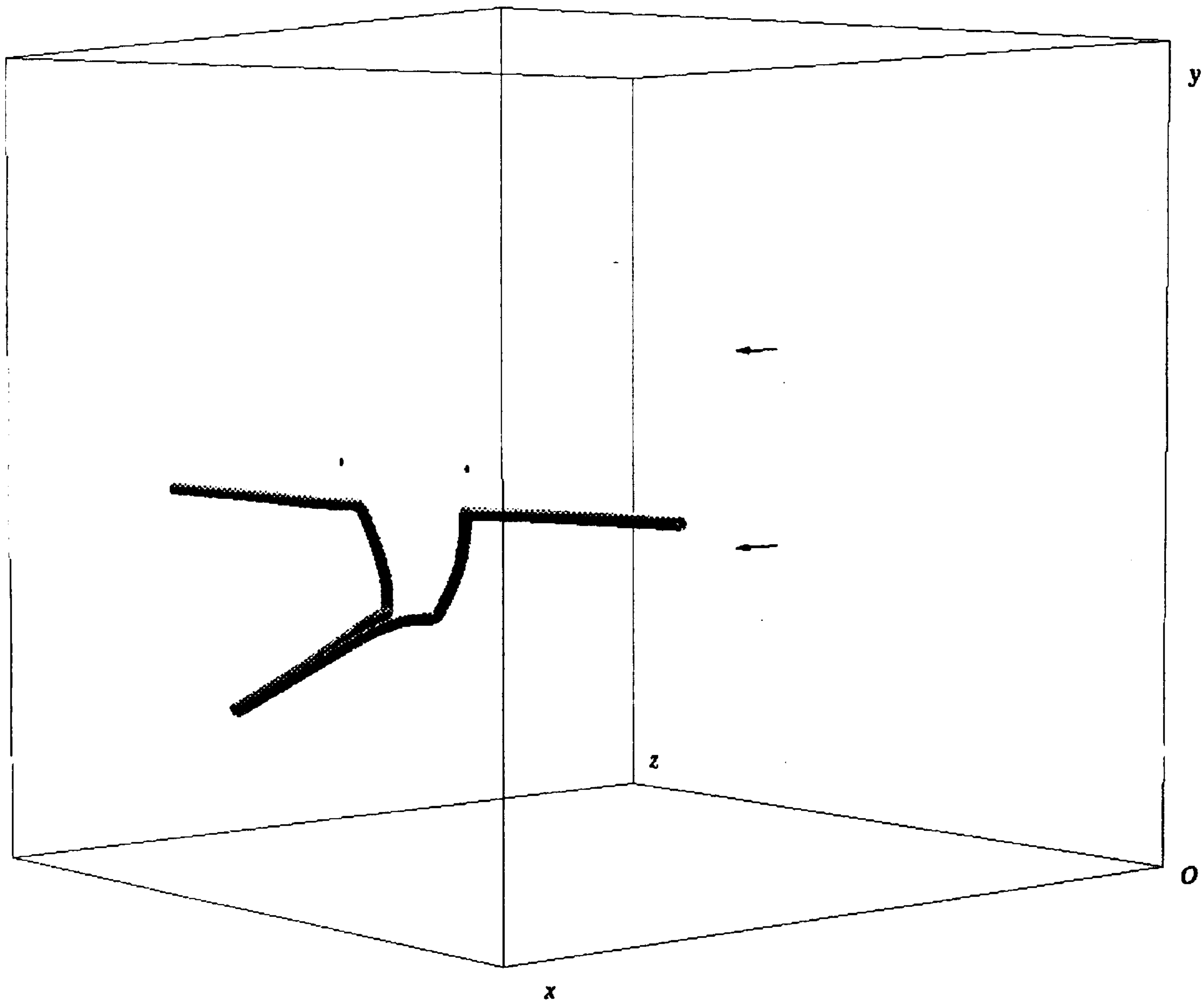

Plate 2

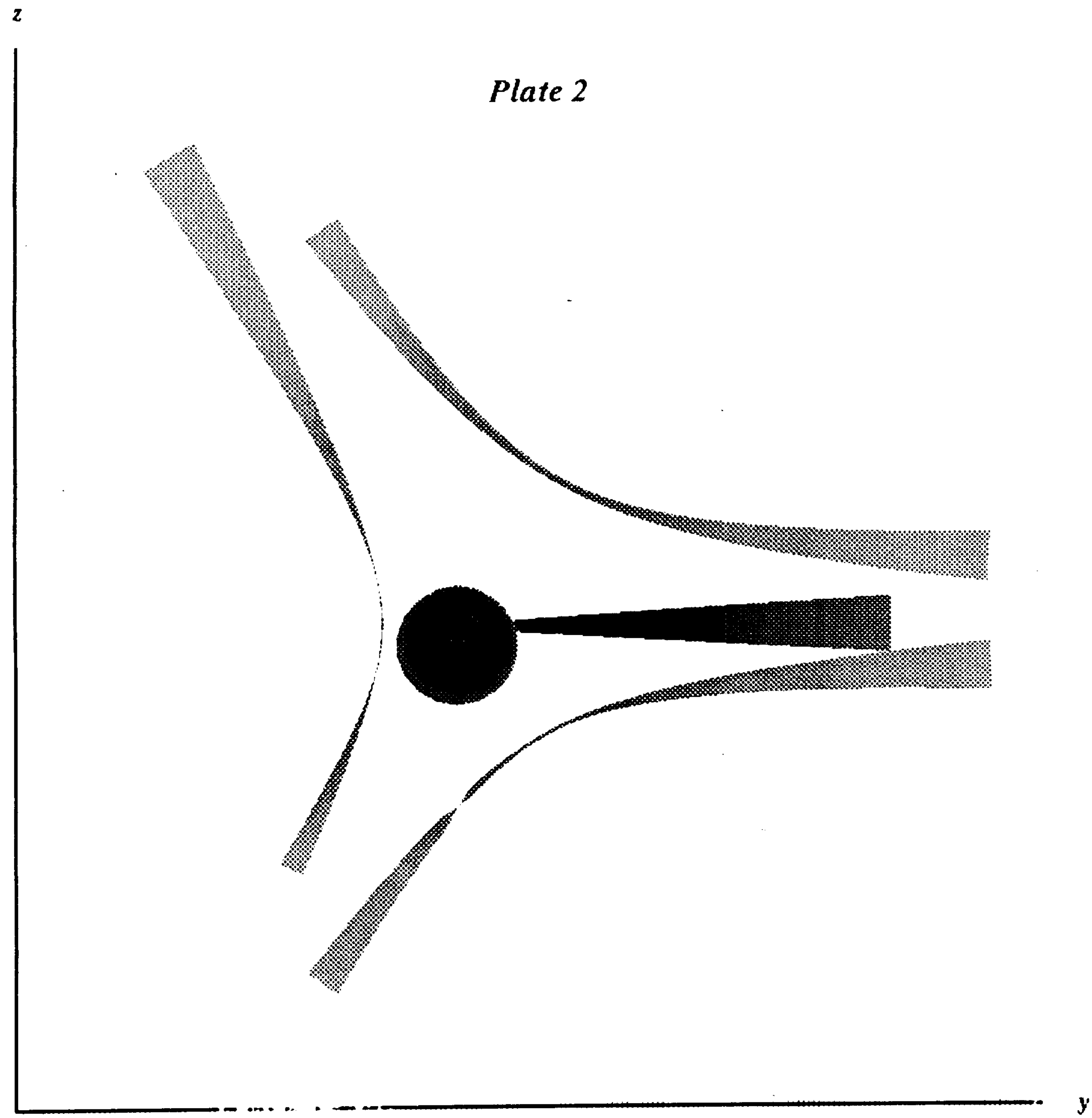

Plate 3

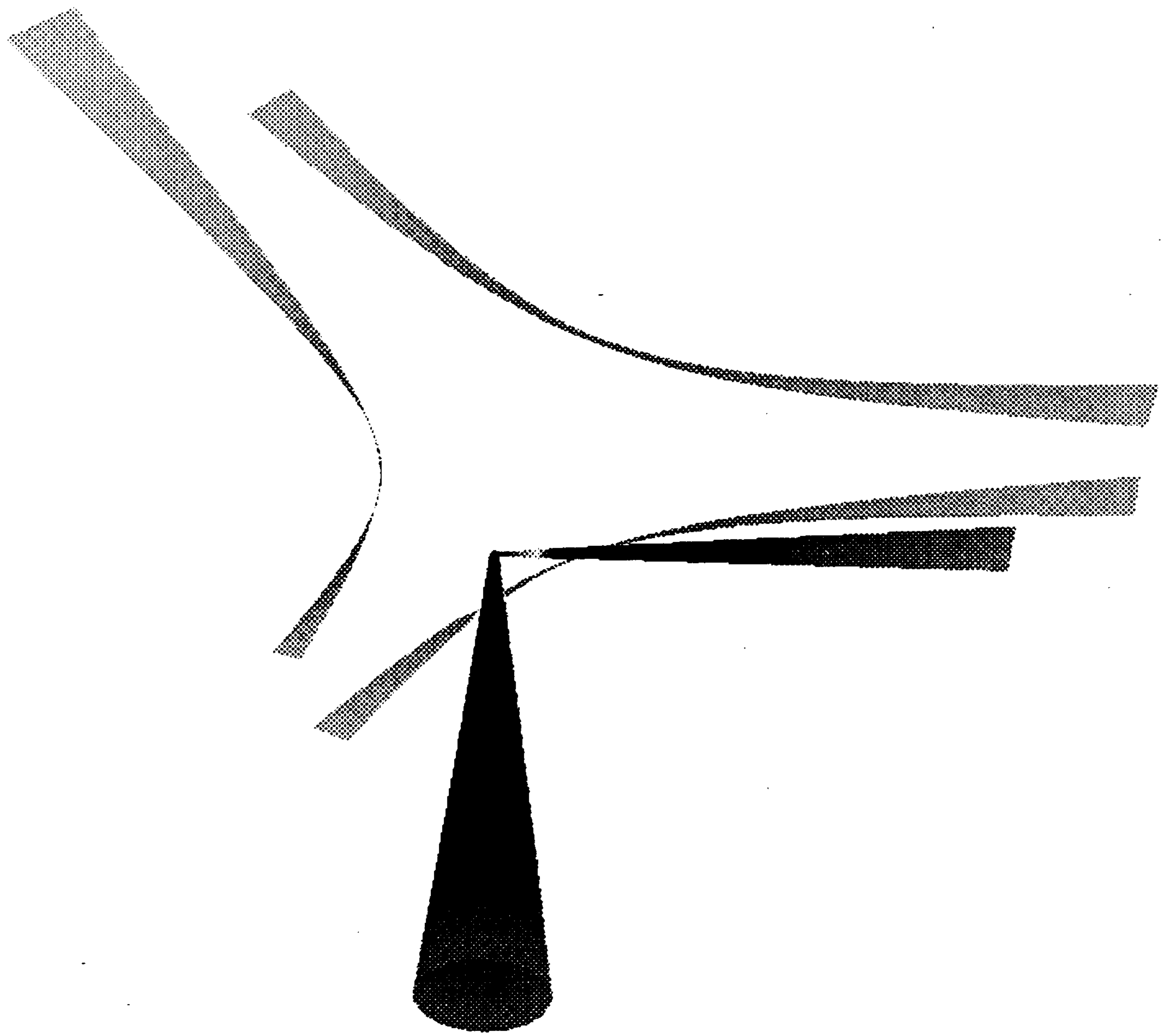

Plate 4